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Two-interval Sturm–Liouville operators in modified Hilbert spaces ☆

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Abstract

By modifying the inner product in the direct sum of the Hilbert spaces associated with each of two underlying intervals on which the Sturm–Liouville equation is defined, we generate self-adjoint realizations for boundary conditions with any real coupling matrix whose determinant is positive. This contrasts with the usual theory which requires the coupling matrix to have determinant one.

Keywords: Self-adjoint; Interface conditions; Transmission conditions

In [1], partly motivated by problems from the applied literature, Everitt and Zettl embarked on a systematic and rigorous study of two-interval Sturm–Liouville problems in the framework of a direct sum of Hilbert spaces. A primary goal of this study was the characterization of *all* self-adjoint realizations in terms of boundary conditions as explicitly as possible. See Chapter 13 in [5] for a detailed exposition of this theory.

Following Mukhtarov and Yakubov [3], in this paper we develop a complete analogue of the Everitt–Zettl theory for regular problems using a direct sum of Hilbert spaces but with the usual inner products replaced by appropriate multiples. The interplay of these multiples with

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the boundary conditions generates self-adjoint problems with arbitrary real coupling matrices K having a positive determinant in contrast with the usual case which requires $\det(K) = 1$.

From another perspective, instead of using multiples of the usual inner products, our approach can be described as using multiples of weight functions.

As in [5] we take the underlying intervals to be open but otherwise arbitrary: they may be identical, disjoint, overlap, abutt; each interval may be the whole real line. On each interval we have a Sturm–Liouville (SL) expression. In the case of identical intervals these expressions may be the same or different. There are four endpoints: the left and right endpoints of each interval. If the intervals abutt, then the right endpoint of the left interval is the same as the left endpoint of the right interval and these "two" endpoints are counted among the four.

1. Notation and basic assumptions

Let

$$J_1 = (a, b), \quad -\infty \leqslant a < b \leqslant \infty, \qquad J_2 = (c, d), \quad -\infty \leqslant c < d \leqslant \infty,$$

and assume the coefficients and weight functions satisfy

$$p_r^{-1}, q_r, w_r \in L(J_r, \mathbb{R}), \quad w_r > 0 \quad \text{a.e. on } J_r, \quad r = 1, 2. \tag{1.1}$$

Define differential expressions M_r by

$$M_r y = -(p_r y')' + q_r y \quad \text{on } J_r,\ r = 1, 2. \tag{1.2}$$

Let

$$H_r = L^2(J_r, w_r).$$

A simple way of getting self-adjoint operators S in the direct sum space

$$H_u = H_1 + H_2, \quad \text{where } H_k = L^2(J_k, w_k),\ k = 1, 2,$$

is to take the direct sum of self-adjoint operators from H_1 and H_2. If these were all the self-adjoint operator realization from the two intervals there would be no need for a "two-interval" theory. As noted in [1] there are many self-adjoint operators which are not merely the sum of self-adjoint operators from each of the separate intervals. These "new" self-adjoint operators involve interactions between the two intervals. It is in these interactions that the new inner products, or the new weight functions, play a role and determine the self-adjoint boundary conditions. Characterizing these interactions as explicitly as possible is our main goal in this paper. This characterization can be given in terms of the values of solutions and their quasi-derivatives at the endpoints and on the multiple inner product parameters. It is well known [5] that, under conditions (1.1), all solutions and their quasi-derivatives have finite limits at each endpoint.

Below we use the notation with a subscript r to denote the rth interval. The subscript r is sometimes omitted when it is clear from the context. For basic facts, notation and terminology see [5].

The two-interval maximal and minimal domains and operators are simply the direct sums of the corresponding one-interval domains and operators:

$$D_{\max} = D_{1\max} + D_{2\max}, \qquad D_{\min} = D_{1\min} + D_{2\min}, \tag{1.3}$$

$$S_{\max} = S_{1\max} + S_{2\max}, \qquad S_{\min} = S_{1\min} + S_{2\min}. \tag{1.4}$$

Elements of $H_u = H_1 + H_2$ will be denoted in bold face type: $\mathbf{f} = \{f_1, f_2\}$ with $f_1 \in H_1$, $f_2 \in H_2$. The standard inner product in H_u is given by

$$(\mathbf{f}, \mathbf{g}) = (f_1, g_1)_1 + (f_2, g_2)_2, \tag{1.5}$$

where $(\cdot,\cdot)_r$ is the usual inner product in H_r:

$$(f_r, g_r)_r = \int\limits_{J_r} f_r \overline{g_r} w_r. \tag{1.6}$$

In this paper, following [3] we replace the direct sum inner product (1.5) by

$$\langle \mathbf{f}, \mathbf{g} \rangle = h(f_1, g_1)_1 + k(f_2, g_2)_2, \quad h > 0,\ k > 0, \tag{1.7}$$

and apply operator theory in the direct sum space

$$H = \bigl(L^2(J_1, w_1) \dotplus L^2(J_2, w_2), \langle\cdot,\cdot\rangle\bigr). \tag{1.8}$$

Remark 1. Note that (1.7) is an inner product in H for any positive numbers h and k. The elements of the Hilbert space H defined by (1.7) are the same as those of the usual direct sum Hilbert space H_u, thus these spaces are differentiated from each other only by their inner products. As we will see below the parameters h, k influence the boundary conditions which yield self-adjoint realizations of the Sturm–Liouville equations in the two-interval case. Observe also that the Hilbert space (1.8) can be viewed as a 'usual' direct sum space H_u with summands $H_r = L^2(J_r, w_r)$ but with each w_r replaced by an appropriate multiple.

As in the one-interval case the Lagrange sesquilinear form is fundamental to the study of boundary value problems. Taking into consideration (1.8) it is defined by

$$[\mathbf{f}, \mathbf{g}] = h[f_1, g_1]_1(b) - h[f_1, g_1]_1(a) + k[f_2, g_2]_2(d) - k[f_2, g_2]_2(c), \tag{1.9}$$

where

$$[f_r, g_r]_r = f_r\bigl(p_r \overline{g_r'}\bigr) - \overline{g_r}\bigl(p_r f_r'\bigr). \tag{1.10}$$

Note that the two-interval Lagrange form $[\mathbf{f}, \mathbf{g}]$ connects all four endpoints with each other and depends on the parameters h, k.

2. Characterization of all self-adjoint extensions

In the one-interval theory the set of all self-adjoint realizations of the SL equation is invariant with respect to the parameter h in the inner product (2.2) below. More specifically we have

Remark 2. The characterization of all self-adjoint realizations of the equation

$$-(py')' + qy = \lambda wy \quad \text{on } J = (a, b),\ -\infty \leqslant a < b \leqslant \infty, \tag{2.1}$$

in $L^2(J, w)$ with $w > 0$ on J in both the regular and singular cases is unchanged when the usual inner product in H is changed to

$$\langle f, g \rangle = h \int\limits_J f \overline{g} w \tag{2.2}$$

for any $h > 0$. In particular, Theorems 10.4.2 through 10.4.10 in [5] hold in the Hilbert space $L^2(J, w)$ with inner product (2.2) for any $h > 0$.

Definition 1. Let the hypotheses and notation of Section 1 hold. By a self-adjoint realization of the equations

$$-(p_r y')' + q_r y = \lambda w_r y \quad \text{on } J_r,\ r = 1, 2, \tag{2.3}$$

in the space $H = (L^2(J_1, w_1) \dot{+} L^2(J_2, w_2), \langle\cdot,\cdot\rangle)$ we mean an operator S from H into H satisfying

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{2.4}$$

From (2.4) it is clear that the self-adjoint realizations are distinguished from each other only by their domains. It is the characterization of *all these* domains explicitly in terms of boundary conditions which is our main goal in this paper. Each operator S satisfying (2.4) can be considered an extension of the minimal operator $S_{\min}$ or, equivalently, a restriction of the maximal operator $S_{\max}$.

First some preliminary lemmas.

Lemma 1. *We have*:

(1) $$S_{\min}^* = S_{1\min}^* \dot{+} S_{2\min}^* = S_{1\max} \dot{+} S_{2\max} = S_{\max},$$
$$S_{\max}^* = S_{1\max}^* \dot{+} S_{2\max}^* = S_{1\min} \dot{+} S_{2\min} = S_{\min}.$$

In particular,

$$D_{\max} = D(S_{\max}) = D(S_{1\max}) \dot{+} D(S_{2\max}),$$

$$D_{\min} = D(S_{\min}) = D(S_{1\min}) \dot{+} D(S_{2\min}).$$

(2) *The minimal operator $S_{\min}$ is a closed, symmetric, densely defined operator in the Hilbert space H with deficiency index $d = 4$.*

Proof. The proof given in [1] for (1.5) extends readily to (1.9). See also Lemma 13.3.1 in [5]. For definitions and discussions of the deficiency index as well as of the one-interval maximal and minimal domains and operators $D(S_{1\max})$, $D(S_{2\max})$, $S_{1\max}$, $S_{2\max}$, $S_{1\min}$, $S_{2\min}$, $D(S_{1\min})$, $D(S_{2\min})$ see [5]. Since the coefficients and the weight function are all real valued the upper and lower deficiency indices are equal and the common value is denoted by d in the two-interval case and by d_1, d_2 for intervals 1 and 2. See the above remark for the effect of replacing the one-interval inner product by a positive multiple of itself. □

We start with the general characterization of the domains of self-adjoint extensions of the two-interval minimal operator. A set of functions $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_d$ from the maximal domain $D_{\max}$ is said to be linearly independent modulo the minimal domain $D_{\min}$ if no nontrivial linear combination of them is in the minimal domain.

Lemma 2. *Let the hypotheses and notation of Section* 1 *hold and let the two-interval minimal and maximal domains $D_{\min}$, $D_{\max}$ and operators $S_{\min}$ and $S_{\max}$ be defined as above. Let the Lagrange form $[\cdot,\cdot]$ be given by* (1.9). *If the operator S with domain $D(S)$, $D_{\min} \subset D(S) \subset D_{\max}$, is a self-adjoint extension of the minimal operator $S_{\min}$, then there exist $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_4 \in D(S) \subset D_{\max}$ satisfying the following conditions*:

(1) $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_4$ *are linearly independent modulo* $D_{\min}$;

(2) $$[\mathbf{g}_j, \mathbf{g}_k] = 0, \quad j, k = 1, 2, \ldots, 4;$$

(3) $$D(S) = \{\mathbf{f} \in D_{\max}\colon\ [\mathbf{f}, \mathbf{g}_j] = 0,\ j = 1, 2, \ldots, 4\}.$$

Conversely, given $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_4 \in D_{\max}$ *satisfying conditions* (1) *and* (2)*, the set* $D(S)$ *defined by* (3) *is a self-adjoint domain.*

Proof. This is an extension of Theorem 10.4.1 in [5] to the multi-interval case. See Theorem 3.1 and Corollary 3.3 in Everitt and Zettl [2] for the case with inner product (1.5); the adaptation to inner product (1.7) is routine. □

Remark 3. Condition (3) is "boundary condition" and conditions (1) and (2) are the conditions on the "boundary conditions" which determine the self-adjoint domains. Condition (1) specifies the number of boundary conditions needed for self-adjointness and condition (2) characterizes the types of these conditions which are self-adjoint.

Remark 4. All three of the conditions of Lemma 2 depend on the maximal domain functions $\mathbf{g}_j$; these depend on the coefficients p_r, q_r and on the weight functions w_r, $r = 1, 2$. This dependence is implicit and complicated. Next we give explicit equivalent conditions for (1)–(3). This can be done by choosing suitable functions $\mathbf{g}_j$ from the maximal domain $D_{\max}$ and using Naimark's Patching Lemma.

Lemma 3 *(The Naimark Patching Lemma). Given any* $c_k \in \mathbb{C}$*,* $k = 1, 2, \ldots, 8$*, there exists a maximal domain function* $\mathbf{g} = \{g_1, g_2\} \in D_{\max}$ *such that*

$$\begin{aligned} &\bar{g}_1(a) = c_1, \qquad \big(p\overline{g_1'}\big)(a) = c_2, \qquad \bar{g}_1(b) = c_3, \qquad \big(p\overline{g_1'}\big)(b) = c_4, \\ &\bar{g}_2(c) = c_5, \qquad \big(p\overline{g_2'}\big)(c) = c_6, \qquad \bar{g}_2(d) = c_7, \qquad \big(p\overline{g_2'}\big)(d) = c_8. \end{aligned} \tag{2.5}$$

Proof. This follows from the one-interval theory, see Lemmas 10.4.1 and 10.4.3 in [5]. □

The next theorem is our main result in this paper; it gives explicit versions of conditions (1)–(3) of Lemma 2.

Theorem 1. *Let the two-interval minimal and maximal domains* $D_{\min}$*,* $D_{\max}$ *and operators* $S_{\min}$ *and* $S_{\max}$ *be defined as above. Let the Lagrange form* $[\cdot,\cdot]$ *be given by* (1.9)*. Then all self-adjoint extensions* S *of the minimal operator* $S_{\min}$ *can be characterized as follows*: *Let* $A = (a_{ij})$*,* $B = (b_{ij})$*,* $C = (c_{ij})$*,* $D = (d_{ij})$ *be* 4 *by* 2 *matrices with complex entries and let* (A, B, C, D) *be the* 4 *by* 8 *matrix whose first two columns are those of* A*, the second two columns are those of* B*, etc. Assume that the following conditions are satisfied*:

(1) *The matrix* (A, B, C, D) *has full rank.*
(2) *For* $i, j = 1, 2, 3, 4$ *we have*

$$\begin{aligned} 0 = {} & k(a_{j1}\bar{a}_{i2} - a_{j2}\bar{a}_{i1}) - k(b_{j1}\bar{b}_{i2} - b_{j2}\bar{b}_{i1}) + h(c_{j1}\bar{c}_{i2} - c_{j2}\bar{c}_{i1}) \\ & - h(d_{j1}\bar{d}_{i2} - d_{j2}\bar{d}_{i1}). \end{aligned} \tag{2.6}$$

The conditions (2.6) *can be expressed more compactly as*

$$kAEA^* - kBEB^* + hCEC^* - hDED^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.7}$$

(3) $D(S)$ *is the set of* $\mathbf{y} = \{y_1, y_2\} \in D_{\max}$ *satisfying*

$$\begin{aligned}
0 &= a_{11}y_1(a) + a_{12}\big(py_1'\big)(a) + b_{11}y_1(b) + b_{12}\big(py_1'\big)(b) \\
&\quad + c_{11}y_2(c) + c_{12}\big(py_2'\big)(c) + d_{11}y_2(d) + d_{12}\big(py_2'\big)(d), \\
0 &= a_{21}y_1(a) + a_{22}\big(py_1'\big)(a) + b_{21}y_1(b) + b_{22}\big(py_1'\big)(b) \\
&\quad + c_{21}y_2(c) + c_{22}\big(py_2'\big)(c) + d_{21}y_2(d) + d_{22}\big(py_2'\big)(d), \\
0 &= a_{31}y_1(a) + a_{32}\big(py_1'\big)(a) + b_{31}y_1(b) + b_{32}\big(py_1'\big)(b) \\
&\quad + c_{31}y_2(c) + c_{32}\big(py_2'\big)(c) + d_{31}y_2(d) + d_{32}\big(py_2'\big)(d), \\
0 &= a_{41}y_1(a) + a_{42}\big(py_1'\big)(a) + b_{41}y_1(b) + b_{42}\big(py_1'\big)(b) \\
&\quad + c_{41}y_2(c) + c_{42}\big(py_2'\big)(c) + d_{41}y_2(d) + d_{42}\big(py_2'\big)(d).
\end{aligned} \tag{2.8}$$

The conditions (2.8) *can be written more compactly as*

$$A\mathbf{Y}_1(a) + B\mathbf{Y}_1(b) + C\mathbf{Y}_2(c) + D\mathbf{Y}_2(d) = 0, \quad \mathbf{Y}_j = \begin{bmatrix} y_j \\ (py_j') \end{bmatrix}, \quad j = 1, 2. \tag{2.9}$$

Proof. The proof consists of applying Theorem 13.3.1, case 5, in Zettl [5] with the weight functions w_1, w_2 replaced by w_1/h, w_2/k, respectively. Note that

$$\operatorname{rank}(A, B, C, D) = \operatorname{rank}(hA, hB, kC, kD)$$

for any $h > 0, k > 0$. Using the Lagrange identity and proceeding as in [5] we note that the introduction of the parameters h, k into the weight functions results in the Lagrange form (1.9) in place of

$$[\mathbf{f}, \mathbf{g}] = [f_1, g_1]_1(b) - [f_1, g_1]_1(a) + [f_2, g_2]_2(d) - [f_2, g_2]_2(c).$$

In terms of Lemma 3, now the proof of Theorem 1 is completed by choosing suitable functions $\mathbf{g}_i = (g_{i1}, g_{i2}) \in D_{\max}$, $i = 1, 2, 3, 4$, such that

$$\begin{aligned}
&\bar{g}_{i1}(a) = -\frac{a_{i2}}{h}, && \big(p\overline{g_{i1}'}\big)(a) = \frac{a_{i1}}{h}, && \bar{g}_{i1}(b) = \frac{b_{i2}}{h}, && \big(p\overline{g_{i1}'}\big)(b) = -\frac{b_{i1}}{h}, \\
&\bar{g}_{i2}(c) = -\frac{c_{i2}}{k}, && \big(p\overline{g_{i2}'}\big)(c) = \frac{c_{i1}}{k}, && \bar{g}_{i2}(d) = \frac{d_{i2}}{k}, && \big(p\overline{g_{i2}'}\big)(d) = -\frac{d_{i1}}{k}.
\end{aligned}$$

See also the proof of Theorem 10.4.2 of [5]—this is the one-interval case and reveals the basic strategy. □

Remark 5. As in the one-interval case, (1) specifies the number of linearly independent conditions, (3) gives the boundary conditions and (2) specifies the conditions on the boundary conditions for self-adjointness. Theorem 1 characterizes all self-adjoint extensions of the two-interval minimal operator or, equivalently, all self-adjoint restrictions of the two-interval maximal operator.

The self-adjointness conditions (2.7) on the boundary conditions involve all four endpoints. By 'splitting' these conditions into separate conditions each for two of the four endpoints we generate a number of corollaries.

Corollary 1. *Let*

$$D(S) = \{\mathbf{y} = \{y_1, y_2\} \in D_{\max}\colon\ A\mathbf{Y}_1(a) + D\mathbf{Y}_2(d) = 0 \text{ and } B\mathbf{Y}_1(b) + C\mathbf{Y}_2(c) = 0\}. \tag{2.10}$$

Assume that $\operatorname{rank}(A, D) = 2$ *and* $kAEA^* - hDED^* = 0$. *Then* $D(S)$ *is a self-adjoint domain in* H *if and only if* $\operatorname{rank}(B, C) = 2$ *and*

$$-kBEB^* + hCEC^* = 0. \tag{2.11}$$

Corollary 2. *Let*

$$D(S) = \{\mathbf{y} = \{y_1, y_2\} \in D_{\max}\colon\ A\mathbf{Y}_1(a) + C\mathbf{Y}_2(c) = 0 \text{ and } B\mathbf{Y}_1(b) + D\mathbf{Y}_2(d) = 0\}. \tag{2.12}$$

Assume that $\operatorname{rank}(A, C) = 2$ *and* $kAEA^* + hCEC^* = 0$. *Then* $D(S)$ *is a self-adjoint domain in* H *if and only if* $\operatorname{rank}(B, D) = 2$ *and*

$$kBEB^* + hDED^* = 0. \tag{2.13}$$

Corollary 3. *Let*

$$D(S) = \{\mathbf{y} = \{y_1, y_2\} \in D_{\max}\colon\ A\mathbf{Y}_1(a) + B\mathbf{Y}_1(b) = 0 \text{ and } C\mathbf{Y}_2(c) + D\mathbf{Y}_2(d) = 0\}. \tag{2.14}$$

Assume that $\operatorname{rank}(A, B) = 2$ *and* $AEA^* - BEB^* = 0$. *Then* $D(S)$ *is a self-adjoint domain in* H *if and only if* $\operatorname{rank}(C, D) = 2$ *and*

$$CEC^* - DED^* = 0. \tag{2.15}$$

Note that these conditions are independent of h *and* k *and are simply the one-interval self-adjointness conditions for each of the two intervals separately. Thus Corollary* 3 *just gives the two-interval self-adjointness conditions which are generated by the direct sum of self-adjoint operators from each of the two intervals separately.*

Remark 6. Note the different signs in Corollary 2. These signs correspond to the signs in the Lagrange form (1.9).

3. Examples

To illustrate Theorem 1 and its corollaries we give a number of examples.

Example 1. Separated boundary conditions at all four endpoints:

$$\begin{aligned}
&A_1 y(a) + A_2(py')(a) = 0, \quad A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0,0),\\
&B_1 y(b) + B_2(py')(b) = 0, \quad B_1, B_2 \in \mathbb{R},\ (B_1, B_2) \neq (0,0),\\
&C_1 y(c) + C_2(py')(c) = 0, \quad C_1, C_2 \in \mathbb{R},\ (C_1, C_2) \neq (0,0),\\
&D_1 y(d) + D_2(py')(d) = 0, \quad D_1, D_2 \in \mathbb{R},\ (D_1, D_2) \neq (0,0).
\end{aligned} \tag{3.1}$$

Let

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ C_1 & C_2 \\ 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ D_1 & D_2 \end{bmatrix}.$$

In this case $\operatorname{rank}(A, D) = 2 = \operatorname{rank}(B, C)$ and

$$0 = AEA^* = BEB^* = CEC^* = DED^*. \tag{3.2}$$

Therefore the conditions of Corollary 1 hold for any h, k.

Example 2. Separated boundary conditions at a and at d and coupled conditions at b, c:

$$\begin{aligned} &A_1 y(a) + A_2(py')(a) = 0, \quad A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0,0), \\ &D_1 y(d) + D_2(py')(d) = 0, \quad D_1, D_2 \in \mathbb{R},\ (D_1, D_2) \neq (0,0). \end{aligned} \tag{3.3}$$

$$Y(c) = KY(b), \quad Y = \begin{bmatrix} y \\ py' \end{bmatrix},\ K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det K > 0. \tag{3.4}$$

Let A, D be as in Example 1, then $\operatorname{rank}(A, D) = 2$ and $kAEA^* - hDED^* = 0$ for any h, k since $0 = AEA^* = DED^*$. Let

$$C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}. \tag{3.5}$$

Then a straightforward computation shows that

$$hCEC^* = kBEB^*$$

is equivalent with

$$hE = k(\det K)E$$

which is equivalent with

$$h = k \det K. \tag{3.6}$$

Therefore, if $h = 1$, and $k > 0$ satisfies $\det K = 1/k$, then the conditions of Corollary 1 hold.
Note that $k > 0$ is needed to preserve the positivity of the weight function w_2/k.

Remark 7. Thus by changing the weight function w_2 to w_2/k we can generate self-adjoint operators for any real coupling matrix K satisfying $\det K = 1/k > 0$. This contrasts with the well-known theory, see Chapter 13 in [5], using the weight function w_2 which requires $\det K = 1$ for self-adjointness. If the boundary conditions are coupled for the endpoint pair a, d as well as the pair b, c then the parameters h, k play a role in both sets of coupled boundary conditions. The next example illustrates this point.

Example 3. Two pairs of coupled conditions:

$$Y(d) = GY(a), \quad G = (g_{ij}),\ g_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det G > 0,$$
$$Y(c) = KY(b), \quad K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det K > 0,\ Y = \begin{bmatrix} y \\ py' \end{bmatrix}. \tag{3.7}$$

Proceeding as in the previous example we obtain the equivalence of the conditions for self-adjointness:

$$kGEG^* = hE \quad \text{and} \quad kKEK^* = hE,$$
$$k \det G = h \quad \text{and} \quad k \det K = h,$$

i.e.,

$$\det G = \det K = \frac{h}{k}.$$

This shows that (3.7) are self-adjoint boundary conditions when positive numbers h, k satisfy $\det G = \det K = h/k$.

Example 4. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ m & -1 \\ 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{3.8}$$

It is easy to check that if $h = k > 0$, then the self-adjointness conditions of Theorem 1 are satisfied for any $m \in \mathbb{R}$. These four matrices yield the boundary conditions

$$y(a) = 0 = y(d), \qquad y(b) = y(c), \qquad (py')(b) - (py')(c) = -my(c). \tag{3.9}$$

Thus, if $b = c$, conditions (3.8) require y to be continuous at $b = c$ but allow the quasi-derivative to have a jump discontinuity at c. If this jump is proportional to the value of y at c with a real proportionality constant $-m$ ($m = 0$ is allowed and reduces to the continuous case) then the jump is self-adjoint. Note that the conditions at a, d are independent of those at c, b and the conditions at a, d can be replaced by any self-adjoint conditions at these two endpoints, i.e., by

$$A_1 E A_1^* = D_1 E D_1^*, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \operatorname{rank}(A_1, D_1) = 2,$$

where A_1, D_1 are 2 by 2 matrices and A, D are the 4 by 2 matrices respectively obtained by inserting two rows of zeros between the two rows of A_1 and between the two rows of D_1.

Example 5. Replacing the matrix C in the previous Example 4 by

$$C = \begin{bmatrix} 0 & 0 \\ -1 & m \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

we get a self-adjoint problem for any real m by choosing $h = k > 0$. When $b = c$ the quasi-derivatives are continuous at b but the solutions are discontinuous when $m \neq 0$. In this case the self-adjoint boundary conditions are

$$y(a) = 0 = y(d), \qquad (py')(b) = (py')(c), \qquad y(b) - y(c) = -m(py')(c).$$

Examples 4 and 5 can be found in [4] where they were established by a completely different method using Green's functions.

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